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Quasispin groups and their generalisations

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Abstract. The generators of the full quasispin group SO(8) and of its subgroup SO(7) are constructed from coupled products of annihilation and creation operators for nucleons. The mutually commuting generators for each of these groups are used to identify the irreducible representations occurring in the nuclear l shell for which $l \leq 3$. The embedding of $SO(8) \times SO(2l+1)$ in $SO(16l+8)$ is examined, and the apparent confluence of spinor and non-spinor representations when $l=0$ is shown to correspond to the well known automorphism for D_4. A quasispin group complementary to G_2 is sought in the nuclear f shell without success. The extension of isospin and spin to an additional spin space leads to the introduction of a unitary symplectic group USp(8) within the new quasispin scheme.

1. Introduction

In the shell theory of electrons or nucleons, it is highly convenient to treat the spin and isospin spaces separately from the orbital space, the limitations on the acceptable states in each of these spaces being determined by applying the Pauli exclusion principle to the combined space. The mathematical way to study the properties of one particular space apart from those of the others is to construct operators from the basic creation and annihilation operators that are scalar in all the spaces except the one in hand. For example, if the tensors $\boldsymbol{a}^\dagger$ and $\boldsymbol{a}$ create and annihilate the $4(2l+1)$ states of an l nucleon, then we can form several coupled products that preserve the number N of nucleons and that are all scalar in the orbital space. Ordering the ranks in the sequence isospin, spin, and orbit, three of these coupled products can be written

$$(\boldsymbol{a}^\dagger \boldsymbol{a})^{(100)}, \qquad (\boldsymbol{a}^\dagger \boldsymbol{a})^{(010)}, \qquad (\boldsymbol{a}^\dagger \boldsymbol{a})^{(110)}. \tag{1}$$

The operators (1) close under commutation and form the generators of the supermultiplet group SU(4) of Wigner (1937). Their application to any state of the nuclear l shell leaves the orbital quantum numbers L and M_L untouched, but allows us to reach all the various spin and isospin states belonging to an irreducible representation of SU(4). The inclusion of the total scalar $(\boldsymbol{a}^\dagger \boldsymbol{a})^{(000)}$ in (1) would have led to U(4) rather than SU(4).

It was realised by Flowers and Szpikowski (1964a, b, 1965) that the collection (1) could be enlarged by including 13 new orbital scalars. Their concise tensorial forms are

$$(\boldsymbol{a}^\dagger \boldsymbol{a}^\dagger)^{(100)}, \qquad (\boldsymbol{a}^\dagger \boldsymbol{a}^\dagger)^{(010)}, \qquad (\boldsymbol{a}\boldsymbol{a})^{(100)}, \qquad (\boldsymbol{a}\boldsymbol{a})^{(010)},$$
$$(\boldsymbol{a}^\dagger \boldsymbol{a})^{(000)} + (\boldsymbol{a}\boldsymbol{a}^\dagger)^{(000)}. \tag{2}$$

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The combined set (1) and (2) closes under commutation and forms the 28 generators of the full quasispin group SO(8). The number N is no longer preserved, and the representations of SO(8) extend across the shell.

The situation for electrons is qualitatively different. The operators $\boldsymbol{a}^\dagger$ and $\boldsymbol{a}$ are double tensors, rather than triple tensors, since the isospin space is no longer present. Coupled products can be formed as before: however we can now use the scalars $(\boldsymbol{a}^\dagger\boldsymbol{a}^\dagger)^{(00)}$ and $(\boldsymbol{aa})^{(00)}$, in conjunction with $(\boldsymbol{a}^\dagger\boldsymbol{a})^{(00)}+(\boldsymbol{aa}^\dagger)^{(00)}$, to form the three components (with suitable prefacing coefficients) Q_+, Q_- and Q_z of the quasispin $\boldsymbol{Q}$. The analogous construction for nuclear states does not work because

$$(\boldsymbol{a}^\dagger\boldsymbol{a}^\dagger)^{(000)}=(\boldsymbol{aa})^{(000)}=0.$$

This result can be readily proved either by explicitly expanding the coupled products or by simply noting that there is no ^{11}S term in the nuclear configuration l^2.

It is the purpose of this article to put these and some allied results in a broad group-theoretical context and to generalise them to hypothetical particles with additional spin spaces.

2. Root figures

Although the full quasispin group SO(8) has been known for almost two decades, there has been very little work on the identification and decomposition of the corresponding irreducible representations that occur in a given nuclear shell. The first step in this direction is to put the generators (1) plus (2) into the standard Cartan–Weyl form. As an aid to doing this we enlarge the collections (1) and (2) to include all operators of the form

$$a_\xi^\dagger a_\eta^\dagger, \qquad a_\xi a_\eta, \qquad a_\xi^\dagger a_\eta \qquad (\xi\neq\eta) \tag{3}$$

and

$$H_\xi=\tfrac{1}{2}[a_\xi^\dagger, a_\xi]=a_\xi^\dagger a_\xi-\tfrac{1}{2}, \tag{4}$$

where ξ and η stand for the triads of quantum numbers of the type (m_t, m_s, m_l). In the atomic case, the isospin space is absent and the operators (3) and (4) form the generators of SO($8l+4$) (Judd 1968). The corresponding group for nucleons is SO($16l+8$), and the states of l^N with even N and odd N span the irreducible representations $(\frac{1}{2}\frac{1}{2}\ldots\frac{1}{2})$ and $(\frac{1}{2}\frac{1}{2}\ldots\frac{1}{2}-\frac{1}{2})$ respectively.

In order to construct the generators of SO(8), we simply form all possible linear combinations of (3) and (4) that are orbital scalars. We write

$$H_{m_t m_s}=\tfrac{1}{2}\sum_{m_l}[a_{m_t m_s m_l}^\dagger, a_{m_t m_s m_l}]$$

for the four operators of type H that commute among themselves. If, now, we form the linear combinations

$$\begin{aligned}
&H_1=\tfrac{1}{2}(H_{\frac{1}{2}\frac{1}{2}}+H_{\frac{1}{2}-\frac{1}{2}}-H_{-\frac{1}{2}\frac{1}{2}}-H_{-\frac{1}{2}-\frac{1}{2}}), \qquad H_2=\tfrac{1}{2}(H_{\frac{1}{2}\frac{1}{2}}-H_{\frac{1}{2}-\frac{1}{2}}-H_{-\frac{1}{2}\frac{1}{2}}+H_{-\frac{1}{2}-\frac{1}{2}}),\\
&H_3=\tfrac{1}{2}(H_{\frac{1}{2}\frac{1}{2}}-H_{\frac{1}{2}-\frac{1}{2}}+H_{-\frac{1}{2}\frac{1}{2}}-H_{-\frac{1}{2}-\frac{1}{2}}), \qquad H_4=\tfrac{1}{2}(H_{\frac{1}{2}\frac{1}{2}}+H_{\frac{1}{2}-\frac{1}{2}}+H_{-\frac{1}{2}\frac{1}{2}}+H_{-\frac{1}{2}-\frac{1}{2}}),
\end{aligned} \tag{5}$$

we find that the roots $(\alpha_1\alpha_2\alpha_3\alpha_4)$ appearing in the equation

$$[H_i, E_\alpha]=\alpha_i E_\alpha \tag{6}$$

are given by

(i) $(\pm1\,0\,0\,1)$, $(0\pm1\,0\,1)$, $(0\,0\pm1\,1)$ for the six operators of type E obtained by forming orbital scalars from $a_\xi^\dagger a_\eta^\dagger$;

(ii) $(\pm1\,0\,0-1)$, $(0\pm1\,0-1)$, $(0\,0\pm1-1)$ for the six E operators obtained from $a_\xi a_\eta$;

(iii) $(\pm1\pm1\,0\,0)$, $(\pm1\,0\pm1\,0)$, $(0\pm1\pm1\,0)$ for the twelve E operators obtained from $a_\xi^\dagger a_\eta$ (with $\xi\neq\eta$).

Thus the root figure corresponds to the 24 possible combinations $\pm\boldsymbol{e}_j\pm\boldsymbol{e}_k$ of the four orthogonal unit vectors $\boldsymbol{e}_i$. We can immediately identify the Cartan algebra as D_4 and the corresponding group as SO(8).

If, now, we discard $a_\xi^\dagger a_\eta^\dagger$ and $a_\xi a_\eta$ from the generators of SO(8), we are left with the number-preserving operators. However, the corresponding group is not semisimple because H_4 commutes with all of them. The removal of H_4 leads to the suppression of the fourth entry α_4 in the weights $(\alpha_1\alpha_2\alpha_3\alpha_4)$. We again get a collection of roots of the type $\pm\boldsymbol{e}_j\pm\boldsymbol{e}_k$, but the weight space is now three-dimensional. The Cartan algebra is D_3 and the corresponding group is SO(6). The isomorphism $D_3\equiv A_3$ ensures the local isomorphism of SO(6) with the Wigner supermultiplet group SU(4).

Different but essentially equivalent identifications of the generators of SO(8) and SO(6) in terms of infinitesimal rotation operators have been made by Flowers and Szpikowski (1964a, b), Pang (1969) and Evans *et al* (1981).

3. The group SO(7)

The relation $SO(8)\supset SO(6)$ makes it attractive to consider the sequence $SO(8)\supset SO(7)\supset SO(6)$. However, the simple insertion of an orthogonal group in seven dimensions, which appears to have been first considered by Pang (1969), cannot automatically be made, since we do not know if the SO(6) group under study falls in the canonical sequence $SO(n)\supset SO(n-1)\supset SO(n-2)\ldots$. For example, Racah (1949) uses $SO(7)\supset SO(3)$ for the study of f electrons, but we obviously cannot insert SO(5) because the irreducible representation (100) of SO(7) yields an f state in SO(3), and there is no seven-dimensional irreducible representation of SO(5).

It is therefore worthwhile to construct the generators of SO(7) explicitly to verify its insertion between SO(8) and SO(6). Since the corresponding root figure (namely B_3) is three-dimensional, we can simply take the three commuting operators H_1, H_2 and H_3 to serve as the H operators for SO(7). The E operators are not quite so easy to identify, but a knowledge that they must be linear combinations of $a_\xi^\dagger a_\eta^\dagger$ and $a_\xi a_\eta$ limits the options open to us. We find that, in addition to the operators for SO(6), we need to include the two vectors

$$\begin{aligned}\boldsymbol{U}&=[(2l+1)/2]^{1/2}[\cos\theta\,(\boldsymbol{a}^\dagger\boldsymbol{a}^\dagger)^{(100)}-\sin\theta\,(\boldsymbol{aa})^{(100)}],\\ \boldsymbol{V}&=[(2l+1)/2]^{1/2}[\cos\theta\,(\boldsymbol{a}^\dagger\boldsymbol{a}^\dagger)^{(010)}+\sin\theta\,(\boldsymbol{aa})^{(010)}].\end{aligned}\tag{7}$$

The eigenvalues $(\alpha_1\,\alpha_2\,\alpha_3)$ of $(H_1\,H_2\,H_3)$ for the components $U_\pm$, $V_\pm$, $(\frac{1}{2})^{1/2}(V_z\pm U_z)$ are $(\pm1\,0\,0)$, $(0\,0\pm1)$, $(0\pm1\,0)$. Thus, to the vectors $\pm\boldsymbol{e}_j\pm\boldsymbol{e}_k$ we must add $\pm\boldsymbol{e}_j$, and this structure corresponds to B_3.

The appearance of the parameter θ in equations (7) is rather surprising, since it means that there are an infinity of ways of specifying SO(7). The transformations (7) bear a resemblance to those of Bogoliubov (see, for example, Lane 1964), the difference being that the creation and annihilation operators appear as coupled pairs rather than singly. The metric tensor $g_{\alpha\beta}$ can be readily calculated, and we find that its determinant contains $\sin 2\theta$ as a factor. Thus, when $\theta = n\pi/2$ (where n is a positive or negative integer, including zero), the determinant vanishes and the group is no longer semi-simple. This result can be verified directly from equations (7), since the vanishing of $\sin 2\theta$ corresponds to the presence of either a pair of creation operators and no annihilation counterpart, or *vice versa*.

4. Irreducible representations

The weights of various many-nucleon states can be found by working out the eigenvalues of the operators H_i. Consider, for example, the configurations s^N. In the case of SO(8) we see at once that, for s^0,

$$(H_1, H_2, H_3, H_4)|0\rangle = (0, 0, 0, -1)|0\rangle.$$

For the state of s^1 for which $m_t = m_s = \frac{1}{2}$, we obtain

$$(H_1, H_2, H_3, H_4)a^\dagger_{\frac{1}{2}\frac{1}{2}}|0\rangle = (\tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, -\tfrac{1}{2})a^\dagger_{\frac{1}{2}\frac{1}{2}}|0\rangle.$$

The other three states of s^1 and the four states of s^3 yield the weights $(\pm\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2})$ with an odd number of minus signs. Continuing, we find

$$(H_1, H_2, H_3, H_4)a^\dagger_{\frac{1}{2}\frac{1}{2}}a^\dagger_{\frac{1}{2}-\frac{1}{2}}|0\rangle = (1, 0, 0, 0)a^\dagger_{\frac{1}{2}\frac{1}{2}}a^\dagger_{\frac{1}{2}-\frac{1}{2}}|0\rangle,$$

while the remaining five states of s^2 yield $(-1, 0, 0, 0)$, $(0, \pm1, 0, 0)$ and $(0, 0, \pm1, 0)$. Specifying an irreducible representation of SO(8) by its highest weight, we see that the states of s^0, s^2 and s^4 belong to the eight-dimensional irreducible representation (1000) of SO(8), while the states of s and s^3 belong to the different eight-dimensional irreducible representation $(\frac{1}{2}\frac{1}{2}\frac{1}{2}-\frac{1}{2})$ of SO(8). From the Weyl (or kaleidoscope) group for D_4, we see that all the listed weights are necessary to complete the structure of an irreducible representation, and there are no weights left over.

Since we have already determined the H operators of SO(7) and SO(6), we can easily determine the branching rules for the irreducible representations of low dimensionality occurring in SO(8) ⊃ SO(7) ⊃ SO(6) simply by deleting the eigenvalues of H_4 from the sequences above. The results are

$$\begin{aligned}&(1000) \to (100)' + (000)' \to [(100) + (000)] + [(000)],\\&(\tfrac{1}{2}\tfrac{1}{2}\tfrac{1}{2}-\tfrac{1}{2}) \to (\tfrac{1}{2}\tfrac{1}{2}\tfrac{1}{2})' \to (\tfrac{1}{2}\tfrac{1}{2}\tfrac{1}{2}) + (\tfrac{1}{2}\tfrac{1}{2}-\tfrac{1}{2}).\end{aligned} \tag{8}$$

Primes are attached to the irreducible representations of SO(7) to distinguish them from those of SO(6). Explicit expressions for the dimensions of the irreducible representations enable various dimension checks to be made. For example, the dimension of $(w_1w_2w_3w_4)$ of SO(8) is given by

$$\begin{aligned}D(w_1w_2w_3w_4) = {} & (w_1 - w_2 + 1)(w_1 - w_3 + 2)(w_1 - w_4 + 3)(w_2 - w_3 + 1)\\&\times(w_2 - w_4 + 2)(w_3 - w_4 + 1)(w_1 + w_2 + 5)(w_1 + w_3 + 4)\\&\times(w_1 + w_4 + 3)(w_2 + w_3 + 3)(w_2 + w_4 + 2)(w_3 + w_4 + 1)/4320.\end{aligned}$$

Corresponding expressions for $D(w_1w_2w_3)$ for SO(7) and SO(6) are given elsewhere (Judd 1963).

5. The nuclear l shell

Since the generators of SO(8) are scalar in orbital space, they necessarily commute with $(\boldsymbol{a}^{\dagger}\boldsymbol{a})^{(001)}$. In fact, they commute with all $(\boldsymbol{a}^{\dagger}\boldsymbol{a})^{(00k)}$ for which k is odd, as can be verified by an explicit calculation. The operators $(\boldsymbol{a}^{\dagger}\boldsymbol{a})^{(00k)}$ with odd k form the generators of $SO(2l+1)$, so we can write

$$SO(16l+8) \supset SO(8) \times SO(2l+1). \tag{9}$$

For the s nucleons of § 4, all orbital states are S states and $SO(2l+1)$ carries no new information. For p nucleons, on the other hand, $SO(2l+1) = SO(3)$, and each irreducible representation of SO(8) is associated with an angular-momentum quantum number L. The precise connection can be worked out by picking states of p^N with $M_L = L$ and finding the eigenvalues of the H_i. The results are set out in table 1. The

Table 1. Branching rules for the states of the nuclear p shell.

Groups and representations[a]		Dimensions
SO(8) × SO(3) $(w_1w_2w_3w_4)L$	SO(8) × SO(3) $(w_1w_2w_3w_4)L$	$D(w_1w_2w_3w_4) \times (2L+1)$
(3000)S	$(\frac{3}{2}\frac{3}{2}\frac{3}{2}-\frac{3}{2})$S	112 × 1
(211 − 1)P	$(\frac{5}{2}\frac{1}{2}\frac{1}{2}-\frac{1}{2})$P	224 × 3
(2100)D	$(\frac{3}{2}\frac{3}{2}\frac{1}{2}-\frac{1}{2})$D	160 × 5
(1110)F	$(\frac{3}{2}\frac{1}{2}\frac{1}{2}\frac{1}{2})$F	56 × 7
(1000)G	$(\frac{1}{2}\frac{1}{2}\frac{1}{2}-\frac{1}{2})$G	8 × 9

[a] The collection of representations $(w_1w_2w_3w_4)L$ in the first and second columns belong to the respective irreducible representations $(\frac{1}{2}\frac{1}{2}\ldots\frac{1}{2})$ and $(\frac{1}{2}\frac{1}{2}\ldots\frac{1}{2}-\frac{1}{2})$ of SO(24). In tables 2 and 3, the corresponding groups are SO(32) and SO(40).

generalisation to d and f nucleons is straightforward, and the results are given in tables 2 and 3. The two irreducible representations $(\frac{1}{2}\frac{1}{2}\ldots\frac{1}{2}\pm\frac{1}{2})$ of $SO(16l+8)$ correspond to configurations with even N (upper sign) and odd N (lower sign). The sum of the listed dimensions for a given l is in each case 2^{8l+4}. This is precisely the number of states in the nuclear l shell.

6. Automorphisms of SO(8)

In the case of s nucleons, the group $SO(16l+8)$ becomes SO(8) itself. It is here that we face an unusual situation. The two representations $(\frac{1}{2}\frac{1}{2}\frac{1}{2}\pm\frac{1}{2})$ of $SO(16l+8)$ for even N and odd N do not coincide with (1000) and $(\frac{1}{2}\frac{1}{2}\frac{1}{2}-\frac{1}{2})$ appearing in (8). The origin of the discrepancy lies in the transformation (5); for, although the eigenvalues of $H_{m_tm_s}$ are always half-integral, those of the H_i can be integral. It is not difficult to see that the eight weights of (1000) and the eight weights of either $(\frac{1}{2}\frac{1}{2}\frac{1}{2}\frac{1}{2})$ or $(\frac{1}{2}\frac{1}{2}\frac{1}{2}-\frac{1}{2})$ are similarly situated with respect to the 24 roots $\pm\boldsymbol{e}_k \pm \boldsymbol{e}_l$ of D_4: all weights are unit distance from the origin and the combined array of roots and the vectors leading from the origin to the eight weights of a representation differ only in their orientation. This is related to the well known automorphism for D_4 (see Wybourne 1974), and leads

Table 2. Branching rules for the states of the nuclear d shell.

Groups and representations		Dimensions
SO(8) × SO(5)	SO(8) × SO(5)	
$(w_1w_2w_3w_4)\times(w'_1w'_2)$	$(w_1w_2w_3w_4)\times(w'_1w'_2)$	$D(w_1w_2w_3w_4)\times D(w'_1w'_2)$
(5000) × (00)	$(\frac{5}{2}\frac{5}{2}\frac{5}{2}-\frac{5}{2})\times(00)$	672 × 1
(322 − 2) × (10)	$(\frac{9}{2}\frac{1}{2}\frac{1}{2}-\frac{1}{2})\times(10)$	1680 × 5
(411 − 1) × (11)	$(\frac{7}{2}\frac{3}{2}\frac{3}{2}-\frac{3}{2})\times(11)$	2400 × 10
(4100) × (20)	$(\frac{5}{2}\frac{5}{2}\frac{3}{2}-\frac{3}{2})\times(20)$	1568 × 14
(321 − 1) × (21)	$(\frac{7}{2}\frac{3}{2}\frac{1}{2}-\frac{1}{2})\times(21)$	2800 × 35
(3200) × (22)	$(\frac{5}{2}\frac{5}{2}\frac{1}{2}-\frac{1}{2})\times(22)$	1400 × 35
(222 − 1) × (30)	$(\frac{7}{2}\frac{1}{2}\frac{1}{2}\frac{1}{2})\times(30)$	672 × 30
(3110) × (31)	$(\frac{5}{2}\frac{3}{2}\frac{3}{2}-\frac{1}{2})\times(31)$	1296 × 81
(2210) × (32)	$(\frac{5}{2}\frac{3}{2}\frac{1}{2}\frac{1}{2})\times(32)$	840 × 105
(2111) × (33)	$(\frac{3}{2}\frac{3}{2}\frac{3}{2}\frac{1}{2})\times(33)$	224 × 84
(3000) × (40)	$(\frac{3}{2}\frac{3}{2}\frac{3}{2}-\frac{3}{2})\times(40)$	112 × 55
(211 − 1) × (41)	$(\frac{5}{2}\frac{1}{2}\frac{1}{2}-\frac{1}{2})\times(41)$	224 × 154
(2100) × (42)	$(\frac{3}{2}\frac{3}{2}\frac{1}{2}-\frac{1}{2})\times(42)$	160 × 220
(1110) × (43)	$(\frac{3}{2}\frac{1}{2}\frac{1}{2}\frac{1}{2})\times(43)$	56 × 231
(1000) × (44)	$(\frac{1}{2}\frac{1}{2}\frac{1}{2}-\frac{1}{2})\times(44)$	8 × 165

to distinct irreducible representations of SO(8) often sharing a common dimensionality. Many examples of this appear in tables 1–3. We can regard the equations (5) as representing a rotation in the four-dimensional weight space in which the star of roots $\pm e_j \pm e_k$ goes into itself while the three elementary representations of dimension 8 transform among themselves according to

$$(\tfrac{1}{2}\tfrac{1}{2}\tfrac{1}{2}\tfrac{1}{2})\rightarrow(1000),\qquad (\tfrac{1}{2}\tfrac{1}{2}\tfrac{1}{2}-\tfrac{1}{2})\rightarrow(\tfrac{1}{2}\tfrac{1}{2}\tfrac{1}{2}-\tfrac{1}{2}),\qquad (1000)\rightarrow(\tfrac{1}{2}\tfrac{1}{2}\tfrac{1}{2}\tfrac{1}{2}).$$

7. Branching rules

It is straightforward to generalise the reductions (8). The branching rules SO(8) → SO(7) and SO(7) → SO(6) are set out in tables 4 and 5 for the nuclear p shell. The weights of the irreducible representations of SO(6) can be interpreted as the weights of the irreducible representations of SU(4), since $D_3 \equiv A_3$. The commuting generators of U(4) are

$$H'_{m_t m_s} = \sum_{m_l} a^{\dagger}_{m_t m_s m_l} a_{m_t m_s m_l},$$

so we can relate them to the commuting generators H_1, H_2 and H_3 of SO(6) by using (5) together with the equation $H_4 = \frac{1}{2}N - 2l - 1$. The results are

$$H'_{\frac{1}{2}\frac{1}{2}} = \tfrac{1}{2}(H_1 + H_2 + H_3) + \tfrac{1}{4}N, \qquad H'_{\frac{1}{2}-\frac{1}{2}} = \tfrac{1}{2}(H_1 - H_2 - H_3) + \tfrac{1}{4}N,$$

$$H'_{-\frac{1}{2}\frac{1}{2}} = \tfrac{1}{2}(-H_1 - H_2 + H_3) + \tfrac{1}{4}N, \qquad H'_{-\frac{1}{2}-\frac{1}{2}} = \tfrac{1}{2}(-H_1 + H_2 - H_3) + \tfrac{1}{4}N.$$

Since the eigenvalues of (H_1, H_2, H_3) are (w_1, w_2, w_3), we can find a characteristic set of eigenvalues for the $H'_{m_t m_s}$ to within the additive constant $\frac{1}{4}N$. A permutation of

Table 3. Branching rules for the states of the nuclear f shell.

Groups and representations SO(8)×SO(7) $(w_1w_2w_3w_4)\times(w'_1w'_2w'_3)$	SO(8)×SO(7) $(w_1w_2w_3w_4)\times(w'_1w'_2w'_3)$	Dimensions $D(w_1w_2w_3w_4)\times D(w'_1w'_2w'_3)$
$(7000)\times(000)$	$(\frac{7}{2}\frac{7}{2}\frac{7}{2}-\frac{7}{2})\times(000)$	$2\,640\times 1$
$(433-3)\times(100)$	$(\frac{13}{2}\frac{1}{2}\frac{1}{2}-\frac{1}{2})\times(100)$	$7\,392\times 7$
$(611-1)\times(110)$	$(\frac{9}{2}\frac{5}{2}\frac{5}{2}-\frac{5}{2})\times(110)$	$12\,320\times 21$
$(522-2)\times(111)$	$(\frac{11}{2}\frac{3}{2}\frac{3}{2}-\frac{3}{2})\times(111)$	$15\,400\times 35$
$(6100)\times(200)$	$(\frac{7}{2}\frac{7}{2}\frac{5}{2}-\frac{5}{2})\times(200)$	$7\,776\times 27$
$(432-2)\times(210)$	$(\frac{11}{2}\frac{3}{2}\frac{1}{2}-\frac{1}{2})\times(210)$	$16\,800\times 105$
$(521-1)\times(211)$	$(\frac{9}{2}\frac{5}{2}\frac{3}{2}-\frac{3}{2})\times(211)$	$22\,680\times 189$
$(5200)\times(220)$	$(\frac{7}{2}\frac{7}{2}\frac{3}{2}-\frac{3}{2})\times(220)$	$10\,752\times 168$
$(431-1)\times(221)$	$(\frac{9}{2}\frac{5}{2}\frac{1}{2}-\frac{1}{2})\times(221)$	$18\,144\times 378$
$(4300)\times(222)$	$(\frac{7}{2}\frac{7}{2}\frac{1}{2}-\frac{1}{2})\times(222)$	$7\,840\times 294$
$(333-2)\times(300)$	$(\frac{11}{2}\frac{1}{2}\frac{1}{2}\frac{1}{2})\times(300)$	$3\,696\times 77$
$(5110)\times(310)$	$(\frac{7}{2}\frac{5}{2}\frac{5}{2}-\frac{3}{2})\times(310)$	$8\,800\times 330$
$(422-1)\times(311)$	$(\frac{9}{2}\frac{3}{2}\frac{3}{2}-\frac{1}{2})\times(311)$	$12\,320\times 616$
$(332-1)\times(320)$	$(\frac{9}{2}\frac{3}{2}\frac{1}{2}\frac{1}{2})\times(320)$	$7\,392\times 693$
$(4210)\times(321)$	$(\frac{7}{2}\frac{5}{2}\frac{3}{2}-\frac{1}{2})\times(321)$	$12\,936\times 1617$
$(3310)\times(322)$	$(\frac{7}{2}\frac{5}{2}\frac{1}{2}\frac{1}{2})\times(322)$	$6\,160\times 1386$
$(4111)\times(330)$	$(\frac{5}{2}\frac{5}{2}\frac{5}{2}-\frac{1}{2})\times(330)$	$2\,400\times 825$
$(3220)\times(331)$	$(\frac{7}{2}\frac{3}{2}\frac{3}{2}\frac{1}{2})\times(331)$	$4\,536\times 2079$
$(3211)\times(332)$	$(\frac{5}{2}\frac{5}{2}\frac{3}{2}\frac{1}{2})\times(332)$	$2\,800\times 2310$
$(2221)\times(333)$	$(\frac{5}{2}\frac{3}{2}\frac{3}{2}\frac{3}{2})\times(333)$	672×1386
$(5000)\times(400)$	$(\frac{5}{2}\frac{5}{2}\frac{5}{2}-\frac{5}{2})\times(400)$	672×182
$(322-2)\times(410)$	$(\frac{9}{2}\frac{1}{2}\frac{1}{2}-\frac{1}{2})\times(410)$	$1\,680\times 819$
$(411-1)\times(411)$	$(\frac{7}{2}\frac{3}{2}\frac{3}{2}-\frac{3}{2})\times(411)$	$2\,400\times 1560$
$(4100)\times(420)$	$(\frac{5}{2}\frac{5}{2}\frac{3}{2}-\frac{3}{2})\times(420)$	$1\,568\times 1911$
$(321-1)\times(421)$	$(\frac{7}{2}\frac{3}{2}\frac{1}{2}-\frac{1}{2})\times(421)$	$2\,800\times 4550$
$(3200)\times(422)$	$(\frac{5}{2}\frac{5}{2}\frac{1}{2}-\frac{1}{2})\times(422)$	$1\,400\times 4095$
$(222-1)\times(430)$	$(\frac{7}{2}\frac{1}{2}\frac{1}{2}\frac{1}{2})\times(430)$	672×3003
$(3110)\times(431)$	$(\frac{5}{2}\frac{3}{2}\frac{3}{2}-\frac{1}{2})\times(431)$	$1\,296\times 7722$
$(2210)\times(432)$	$(\frac{5}{2}\frac{3}{2}\frac{1}{2}\frac{1}{2})\times(432)$	840×9009
$(2111)\times(433)$	$(\frac{3}{2}\frac{3}{2}\frac{3}{2}\frac{1}{2})\times(433)$	224×6006
$(3000)\times(440)$	$(\frac{3}{2}\frac{3}{2}\frac{3}{2}-\frac{3}{2})\times(440)$	112×3003
$(211-1)\times(441)$	$(\frac{5}{2}\frac{1}{2}\frac{1}{2}-\frac{1}{2})\times(441)$	224×8008
$(2100)\times(442)$	$(\frac{3}{2}\frac{3}{2}\frac{1}{2}-\frac{1}{2})\times(442)$	160×10296
$(1110)\times(443)$	$(\frac{3}{2}\frac{1}{2}\frac{1}{2}\frac{1}{2})\times(443)$	56×9009
$(1000)\times(444)$	$(\frac{1}{2}\frac{1}{2}\frac{1}{2}-\frac{1}{2})\times(444)$	8×4719

these numbers is enough to determine the highest weight and hence the corresponding tableau $[\lambda]$ of U(4). The results are displayed in table 6 for those representations occurring in the nuclear p shell. The tableaux separated by commas in this table correspond to the various choices of N open to us and become equivalent when U(4) is limited to its subgroup SU(4).

Table 4. Branching rules for SO(8) → SO(7).

$(w_1w_2w_3w_4)$	$(w_1w_2w_3)'$
$(\frac{1}{2}\frac{1}{2}\frac{1}{2}\pm\frac{1}{2})$	$(\frac{1}{2}\frac{1}{2}\frac{1}{2})'$
$(\frac{3}{2}\frac{1}{2}\frac{1}{2}\pm\frac{1}{2})$	$(\frac{3}{2}\frac{1}{2}\frac{1}{2})'(\frac{1}{2}\frac{1}{2}\frac{1}{2})'$
$(\frac{3}{2}\frac{3}{2}\frac{1}{2}\pm\frac{1}{2})$	$(\frac{3}{2}\frac{3}{2}\frac{1}{2})'(\frac{3}{2}\frac{1}{2}\frac{1}{2})'$
$(\frac{3}{2}\frac{3}{2}\frac{3}{2}\pm\frac{3}{2})$	$(\frac{3}{2}\frac{3}{2}\frac{3}{2})'$
$(\frac{5}{2}\frac{1}{2}\frac{1}{2}\pm\frac{1}{2})$	$(\frac{5}{2}\frac{1}{2}\frac{1}{2})'(\frac{3}{2}\frac{1}{2}\frac{1}{2})'(\frac{1}{2}\frac{1}{2}\frac{1}{2})'$
(1000)	(100)′(000)′
(1110)	(111)′(110)′
(2100)	(210)′(200)′(110)′(100)′
(211 ± 1)	(211)′(111)′
(3000)	(300)′(200)′(100)′(000)′

Table 5. Branching rules for SO(7) → SO(6).

$D(w_1w_2w_3)'$	$(w_1w_2w_3)'$	$(w_1w_2w_3)$
8	$(\frac{1}{2}\frac{1}{2}\frac{1}{2})'$	$(\frac{1}{2}\frac{1}{2}\frac{1}{2})(\frac{1}{2}\frac{1}{2}-\frac{1}{2})$
48	$(\frac{3}{2}\frac{1}{2}\frac{1}{2})'$	$(\frac{3}{2}\frac{1}{2}\frac{1}{2})(\frac{3}{2}\frac{1}{2}-\frac{1}{2})(\frac{1}{2}\frac{1}{2}\frac{1}{2})(\frac{1}{2}\frac{1}{2}-\frac{1}{2})$
112	$(\frac{3}{2}\frac{3}{2}\frac{1}{2})'$	$(\frac{3}{2}\frac{3}{2}\frac{1}{2})(\frac{3}{2}\frac{3}{2}-\frac{1}{2})(\frac{3}{2}\frac{1}{2}\frac{1}{2})(\frac{3}{2}\frac{1}{2}-\frac{1}{2})$
112	$(\frac{3}{2}\frac{3}{2}\frac{3}{2})'$	$(\frac{3}{2}\frac{3}{2}\frac{3}{2})(\frac{3}{2}\frac{3}{2}-\frac{3}{2})(\frac{3}{2}\frac{3}{2}\frac{1}{2})(\frac{3}{2}\frac{3}{2}-\frac{1}{2})$
168	$(\frac{5}{2}\frac{1}{2}\frac{1}{2})'$	$(\frac{5}{2}\frac{1}{2}\frac{1}{2})(\frac{5}{2}\frac{1}{2}-\frac{1}{2})(\frac{3}{2}\frac{1}{2}\frac{1}{2})(\frac{3}{2}\frac{1}{2}-\frac{1}{2})(\frac{1}{2}\frac{1}{2}\frac{1}{2})(\frac{1}{2}\frac{1}{2}-\frac{1}{2})$
1	(000)′	(000)
7	(100)′	(100)(000)
21	(110)′	(110)(100)
35	(111)′	(111)(11 − 1)(110)
27	(200)′	(200)(100)(000)
105	(210)′	(210)(200)(110)(100)
189	(211)′	(211)(21 − 1)(210)(111)(11 − 1)(110)
77	(300)′	(300)(200)(100)(000)

It is straightforward to confirm that tables 1, 4, 5 and 6 are consistent with the classification of the nuclear p shell provided by Elliott and Lane (1957). For example, the D states occurring in p^N (with even N) belong to (2100) of SO(8), which decomposes, via representations of SO(7), to the direct sum

$$2(000)+4(100)+2(110)+2(200)+(210)$$

of representations of SO(6). From table 6, this sequence is equivalent to

$$2[0]+4[11]+2[211]+2[22]+[321]. \tag{10}$$

Allowing for the single occurrence of [321] in the half-filled p shell, and for the equivalences set out in table 6, the tableaux in the sequence (10) above are just the adjoints of the orbital tableaux listed for the D states in table 18 of Elliott and Lane (1957). We should point out here that many of the decompositions represented by tables 1–6 are known to Dr J A Evans.

Table 6. Correspondence between irreducible representations of SO(6) and U(4).

$D(w_1w_2w_3)$	$(w_1w_2w_3)$	$[\lambda]$
4	$(\frac{1}{2}\frac{1}{2}\frac{1}{2})$	[1], [2111], [3222]
4	$(\frac{1}{2}\frac{1}{2}-\frac{1}{2})$	[111], [2221], [3332]
20	$(\frac{3}{2}\frac{1}{2}\frac{1}{2})$	[21], [3211]
20	$(\frac{3}{2}\frac{1}{2}-\frac{1}{2})$	[221], [3321]
36	$(\frac{3}{2}\frac{3}{2}\frac{1}{2})$	[311]
36	$(\frac{3}{2}\frac{3}{2}-\frac{1}{2})$	[322]
20	$(\frac{3}{2}\frac{3}{2}\frac{3}{2})$	[3]
20	$(\frac{3}{2}\frac{3}{2}-\frac{3}{2})$	[333]
60	$(\frac{5}{2}\frac{1}{2}\frac{1}{2})$	[32]
60	$(\frac{5}{2}\frac{1}{2}-\frac{1}{2})$	[331]
1	(000)	[0], [1111], [2222], [3333]
6	(100)	[11], [2211], [3322]
15	(110)	[211], [3221]
10	(111)	[2], [3111]
10	(11−1)	[222], [3331]
20	(200)	[22], [3311]
64	(210)	[321]
45	(211)	[31]
45	(21−1)	[332]
50	(300)	[33]

The spin-isotopic spin structure of the irreducible representations of SO(6) can be found by working out the decompositions corresponding to SO(6) → SO(3) × SO(3). Many equivalent decompositions have been tabulated by Jahn (1950) and Flowers (1952). The representations of SO(6) are specified in their articles by the equivalent representations of U(4), and those of SO(3) × SO(3) by the quantum-number pairs (TS).

8. Complementary groups

The 160 D states occurring in $p^2, p^4, \ldots, p^{10}$ can be regarded as basis functions for the representation (2100) of SO(8). The properties of the D states can thus be tied to the properties of the group SO(8). This connection parallels the use of seniority in atomic physics, where matrix elements of operators in different configurations can be related by making use of their tensorial ranks with respect to the quasispin $\boldsymbol{Q}$. A similar transference is possible in the nuclear case, though the properties of SO(8) are not as well known or as easily come by as those of SO(3). Moshinsky and Quesne (1969, 1970) have called such groups *complementary* to stress the role that they can play. For example, the reduced matrix elements of an orbital vector operator within the states of p^N with a given L become proportional to certain Clebsch–Gordan coefficients for the group SO(8). Of course, if such coefficients vanish, so do the corresponding matrix elements; but it is rare that new selection rules can be obtained in this way. The richness of the classification scheme becomes more striking for particles for which $l > 1$. For d and f nucleons the orbital group SO(3) of table 1 is replaced by SO(5) and SO(7) respectively. As can be seen from tables 2 and 3, new irreducible representations of SO(8) appear.

For many years it has seemed as if it should be possible to find a group complementary to the group G_2 that Racah (1949) introduced for f electrons. The additional selection rules that such a group would provide might go some way towards accounting for many of the unexpected simplifications that G_2 is associated with (see Judd 1971). Moreover, an obvious starting point in constructing the generators of the complementary group is the collection of triple products of the type $(\boldsymbol{a}^\dagger\boldsymbol{a}^\dagger\boldsymbol{a}^\dagger)^{(\frac{3}{2}0)}$, $(\boldsymbol{a}^\dagger\boldsymbol{a}\boldsymbol{a})^{(\frac{1}{2}0)}$, etc. All such orbital scalars turn out to be also scalar with respect to G_2. Unfortunately we have not succeeded in picking a reasonably small subset of them that closes under commutation.

Table 3 is useful for studying whether the nuclear f shell is more propitious. From the branching rules for $SO(7)\rightarrow G_2$ given by Wybourne (1970), we quickly find that the number of occurrences of the irreducible representations (80), (71), (70), (62), ... of G_2 in the nuclear f shell are 16, 112, 576, 320, ... If a group C complementary to G_2 exists, these numbers (or their halves, if a separation according to even or odd N occurs) should match the dimensions of representations of C. The trivial solution

$$C' = SU(16)\times SU(112)\times SU(576)\times\ldots$$

is of no interest to us, since it adds nothing to our knowledge of the structure of the shell. Instead, we seek a group C for which $C'\supset C\supset SO(8)$. It is easy to check that the numbers 16, 112, 576, 320, ... (or their halves) cannot be the dimensions of *irreducible* representations of a single group C; and without the crucial feature of irreducibility much of the attraction of the analysis evaporates. The splitting of the shell into parts for which $N\equiv p \pmod 3$, where $p=0$, 1, and 2, can be studied with the aid of table 3 above and table 6 of Flowers (1952). It again turns out that the occurrences of irreducible representations of G_2 do not correspond to the dimensions of the irreducible representations of a group C. Thus the obvious avenues leading to a group complementary to G_2 do not appear to be productive.

9. Generalised isospin

For nucleons, $\boldsymbol{a}^\dagger$ and $\boldsymbol{a}$ are triple tensors. If we generalise s and t by including a third spin r, also equal to $\frac{1}{2}$, then $\boldsymbol{a}^\dagger$ and $\boldsymbol{a}$ become quadruple tensors. The coupled products $(\boldsymbol{a}^\dagger\boldsymbol{a}^\dagger)^{(0000)}$ and $(\boldsymbol{a}\boldsymbol{a})^{(0000)}$ are *not* identically zero, and we can construct a quasispin vector $\boldsymbol{Q}$ having components Q_+ and Q_- proportional to these tensors, in analogy with the electron case. It is not difficult to see that this will always be possible when we are dealing with particles with an odd number of spin ranks of $\frac{1}{2}$. The absence of a physical interpretation of r is no reason for not pursuing the implications of the third spin, since we can always project out the nuclear case by taking $m_r=+\frac{1}{2}$. (In the same way we can recover the electron analysis from that of the nucleons by taking $m_t=+\frac{1}{2}$.)

The existence of $\boldsymbol{Q}$ means that we can regard $\boldsymbol{a}^\dagger$ and $\boldsymbol{a}$ as the two components (with $m_q=\frac{1}{2}$ and $-\frac{1}{2}$) of a quasispin tensor $\boldsymbol{a}^{(qrtsl)}$ for which $q=\frac{1}{2}$. To facilitate the discussion, we write

$$\boldsymbol{X}^{(K\kappa_1\kappa_2\kappa_3k)} = (\boldsymbol{a}^{(qrtsl)}\boldsymbol{a}^{(qrtsl)})^{(K\kappa_1\kappa_2\kappa_3k)}. \tag{11}$$

We begin by restricting ourselves to generalisations of the quasispin groups: thus we set $k=0$. The four vectors

$$\boldsymbol{X}^{(10000)}, \qquad \boldsymbol{X}^{(01000)}, \qquad \boldsymbol{X}^{(00100)}, \qquad \boldsymbol{X}^{(00010)} \tag{12}$$

are proportional to the angular momentum vectors $\boldsymbol{Q}$, $\boldsymbol{R}$, $\boldsymbol{T}$ and $\boldsymbol{S}$. Since the components of $\boldsymbol{a}^{(qrtsl)}$ satisfy anticommutation relations, the interchange of the two tensors $\boldsymbol{a}^{(qrtsl)}$ in (11) produces a sign change if

$$2q-K+2r-\kappa_1+2t-\kappa_2+2s-\kappa_3+2l-k \tag{13}$$

is even, and a numerical residue if $K=\kappa_1=\kappa_2=\kappa_3=k=0$. Thus $\boldsymbol{X}^{(K\kappa_1\kappa_2\kappa_3k)}=0$ when the sum of the ranks is even and greater than zero. We can now see that the four tensors

$$\boldsymbol{X}^{(01000)}, \qquad \boldsymbol{X}^{(00100)}, \qquad \boldsymbol{X}^{(00010)}, \qquad \boldsymbol{X}^{(01110)}, \tag{14}$$

together with the operators

$$X_0^{(10000)}, \qquad X_0^{(11100)}, \qquad X_0^{(10110)}, \qquad X_0^{(11010)}, \tag{15}$$

close under commutation and represent all possible coupled pairs of creation and annihilation operators that are scalar in orbital space and that preserve the number of particles. There are 64 components in all and they form the generators of U(8). If we omit $X_0^{(10000)}$, we are left with the 63 generators of SU(8). This group is the generalisation of the Wigner supermultiplet group SU(4). If we permit components of K other than zero in the tensors (15), the augmented collection comprises the 120 generators of SO(16). This is the generalisation of the quasispin group SO(8).

One new feature comes out of this analysis. From the generators of U(8) we can discard the operators (15) and retain just the 36 components (14). This subset closes under commutation and constitutes the generators of the unitary symplectic group USp(8). If we omit the tensor $\boldsymbol{X}^{(01110)}$, we can extend the sequence of groups and subgroups to

$$\mathrm{SO}(16)\supset\mathrm{U}(8)\supset\mathrm{USp}(8)\supset\mathrm{SO}(3)\times\mathrm{SO}(3)\times\mathrm{SO}(3),$$

where the three SO(3) groups on the right have as their generators $\boldsymbol{R}$, $\boldsymbol{T}$ and $\boldsymbol{S}$.

It is not difficult to see that the existence of the SO(7) group in the sequence $\mathrm{SO}(8)\supset\mathrm{SO}(7)\supset\mathrm{SU}(4)$ does not generalise to $\mathrm{SO}(16)\supset\mathrm{X}\supset\mathrm{SU}(8)$, since there is no way that additional roots can be added to the root diagram A_7 (corresponding to SU(8)) to give an acceptable root diagram that is itself contained in D_8 (corresponding to SO(16)). The isomorphism $A_3\equiv D_3$ is a key element in allowing for the intervention of SO(7), and this feature is not susceptible of generalisation.

10. Concluding remarks

Although our general approach in this paper has been somewhat formal, we are now in a position to turn our attention to problems of direct physical interest. For example, it would be interesting to follow up the remarks made in § 8 on complementary groups to see just how the Wigner–Eckart theorem for the full quasispin group SO(8) might be used to relate matrix elements of nuclear operators to one another. The projection technique briefly mentioned in § 9 is another attractive area for study. It is possible that vestigial features of groups such as USp(8) might be retained for nuclear or electronic configurations. If this turns out to be the case, we have a source for additional classificatory symbols and with them the possibility of providing explanations for some of the unexpected simplifications in the patterns of null matrix elements. We cannot yet afford to assume a complacent stance on such matters.

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